Subspace local quantum channels

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Abstract. A special class of quantum channels, named *subspace local* (SL), are defined and investigated. The proposed definition of subspace locality of quantum channels is an attempt to answer the question of what kind of restriction should be put on a channel, if it is to act 'locally' with respect to two 'locations', when these naturally correspond to a separation of the total Hilbert space in an orthogonal sum of subspaces $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, rather than a tensor product decomposition $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. It is shown that the set of SL channels decomposes into four disjoint families of channels. Explicit expressions to generate all channels in each family is presented. It is shown that one of these four families, the *local subspace preserving* (LSP) channels, is precisely the intersection between the set of *subspace preserving* channels and the SL channels. For a subclass of the LSP channels, a special type of unitary representation using ancilla systems is presented.

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1. Introduction

Questions tied to locality is a central theme in many investigations of quantum mechanics, like correlation, entanglement, teleportation and questions about what kind of nonlocal resources are needed in performing various operations. Usually a system consisting of two separate entities on separate locations are modeled with a Hilbert space in form of a tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ represents the pure states of the subsystem at location 1 and correspondingly for $\mathcal{H}_2$. General states of the system are represented by density operators on the Hilbert space. Operations on this bipartite system is represented by 'channels', which are trace preserving completely positive maps (CPMs). Channels map states to states of the system, and give a quite general tool to describe evolution which allows interaction with external quantum systems. A channel Φ can be said to be local with respect to this bipartite system, if it can be written as a product channel $\Phi = \Phi_1 \otimes \Phi_2$, where Φ_1 and Φ_2 act on density operators on the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. In other words the total operation is decomposed into operations which acts only locally on each subsystem. The product channels are precisely those which can be achieved with 'local means' only.

The central assumption in this modeling is that the separation in two locations is associated with a decomposition of the total Hilbert space into a tensor product. However, there are situations where the separation in locations is not directly associated with such a tensor decomposition, but rather a decomposition into two orthogonal subspaces. One example is the two arms of a two path single particle interferometer. The particle can be in either path 1 or path 2 (possibly the particle also carries an internal degree of freedom.) Reasonably, the Hilbert space $\mathcal{H}$ of the particle in the interferometer is decomposed into an orthogonal sum of a Hilbert space $\mathcal{H}_1$ representing the pure states localized in path 1, and a Hilbert space $\mathcal{H}_2$ representing pure states localized in arm 2. Hence $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Clearly the separation into the two locations corresponds to a decomposition of the total Hilbert space into an orthogonal sum.

Suppose we let a channel Φ act on the state of the particle in the interferometer. In other words the particle can be affected in a very general way, possibly interacting with other quantum systems, while passing the arms of the interferometer. What kind of restrictions should be put on Φ if it is supposed to act 'locally' with respect to the two locations? Hence, Φ should be possible to realize using only 'local resources and means' on each location. In other words, what we look for is a reasonable definition of 'local channel', when the separation into locations corresponds to a decomposition into an orthogonal sum of the total Hilbert space.

The *subspace local channels* presented here is an attempt to construct such a definition. Whatever definition one assumes, it should be consistent with the definition of locality with respect to subsystems i.e. separation into tensor product. The strategy used here to define subspace locality, is to embed the original state space in a larger state space, in such a way that the orthogonal sum decomposition is 'transformed' into a tensor decomposition. When having this decomposition in tensor product, the standard definition of locality in terms of product channels can be applied. The embedding is achieved by changing the basic description of the particle state, from being a description of the state the particle, to a description of the occupation states at the two locations, hence an occupation number representation of a second quantization of the original state space. The definition of subspace local channels in terms of these larger spaces is thereafter translated back to the original state space.

The main advantage of this definition is that it is based on the standard definition of product channels, in a way that can be given a physical interpretation, and which can be applied in very general settings. It must however be emphasized that it is not certain that this is the most appropriate definition in all contexts. In this study the consequences of this definition is investigated. Future investigations will have to settle whether or not this, or other alternative definitions, are best suited to fit the intuitive notion of "subspace local" channels.

The investigations performed here are all made under the assumption that the Hilbert spaces involved are finite-dimensional. This restriction is primarily made of practical reasons, to get reasonably rigorous derivations without getting too involved into mathematical technicalities. The author believes that many of the propositions, with suitable technical modifications, remains true for infinite-dimensional separable Hilbert spaces. That question will, however, not be treated here.

The structure of this article is as follows: in section 2 notation and conventions is presented. In section 3 we give the basic definition of subspace locality in terms of second quantization of the state spaces of the systems. It is shown that a large part of these second quantized spaces are irrelevant for the analysis and that the definition can be reformulated on smaller subspaces. In section 4 the local subspace preserving channels (LSP) are defined. In section 4.1 a special type of unitary representation of a subclass of the LSP channels is presented. In section 5 we turn to the set of SL channels and show that this set is partitioned into four families of channels. Explicit formulas to generate all the channels in each of these four families, is deduced. In section 6 we discuss some conceptual aspects of the nature of the SL channels. A summary is given in 7.

2. Notation and conventions

Complex Hilbert spaces are denoted by $\mathcal{H}$ with various subscripts. Given two Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_T$, a CPM ϕ is a linear map from the set of trace-class operators $\tau(\mathcal{H}_S)$ to the set of trace-class operators $\tau(\mathcal{H}_T)$. We say that $\mathcal{H}_S$ is the *source space* of ϕ (or just *source*), and $\mathcal{H}_T$ is the *target space* of ϕ (or just *target*). On finite-dimensional Hilbert spaces, the set of trace class operators coincide with the set of linear operators. We let $\mathcal{L}(\mathcal{H})$ denote the set of all linear operators from the Hilbert space $\mathcal{H}$ to itself and let $\mathcal{L}(\mathcal{H}_S, \mathcal{H}_T)$ denote the set of linear operators from $\mathcal{H}_S$ to $\mathcal{H}_T$. Trace preserving CPMs (channels) are denoted by capital Greek letters, to distinguish them from the general CPMs which are denoted by small Greek letters.

Orthogonal decompositions in pairs of subspaces of both the source and target space, i.e. $\mathcal{H}_S = \mathcal{H}_{s1} \oplus \mathcal{H}_{s2}$ and $\mathcal{H}_T = \mathcal{H}_{t1} \oplus \mathcal{H}_{t2}$, play a central role. These subspaces are assumed to be at least one-dimensional, hence $\mathcal{H}_S$ and $\mathcal{H}_T$ are both at least two-dimensional. To each of these spaces correspond orthogonal projectors and orthonormal bases. Each row of the following table consists of a space, the corresponding projection operator, and the notation for an arbitrary orthonormal basis of the subspace in question:

$$\begin{array}{ccc} \mathcal{H}_S & \mathcal{H}_{s1} & \mathcal{H}_{s2} \\ \hat{1}_S & P_{s1} & P_{s2} \\ \{|S,n\rangle\}_n & \{|s1,k\rangle\}_k & \{|s2,l\rangle\}_l, \end{array}$$

where the index span the appropriate number of elements in each case. Similar notation is used for the spaces $\mathcal{H}_T$, $\mathcal{H}_{t1}$ and $\mathcal{H}_{t2}$. The spaces $\mathcal{H}_T$, $\mathcal{H}_S$ and their subspaces are referred to as the 'first quantized' spaces.

Unfortunately the list of spaces does not end here but continues with second quantized versions of these spaces. $F^x(\mathcal{H})$ denotes the occupation number representation of a second quantization of the Hilbert space $\mathcal{H}$. The index x denotes the type of second quantization (fermionic or bosonic). This choice determines how many particles can occupy a state in the first quantized space. $F_1(\mathcal{H})$ represents the pure single-particle states. $F_0(\mathcal{H})$ is the space spanned by the vacuum state, and $F_{01}(\mathcal{H}) = F_0(\mathcal{H}) \oplus F_1(\mathcal{H})$. These spaces do not carry any index to denote the type of quantization, since these subspaces are independent of that choice. $F_2^x(\mathcal{H})$ represents all pure states with at least two particles. This subspace is affected by the choice of statistics. One may note that $F_1(\mathcal{H}) \simeq \mathcal{H}$ and that $F_0(\mathcal{H}) \simeq \mathbb{C}$. Note moreover that if $\mathcal{H}$ is one-dimensional and if we have chosen fermionic second quantization, then $F_2^x(\mathcal{H})$ is zero-dimensional. The following table shows the notation for the relevant subspaces of the second quantization of the source space.

$$\begin{array}{cccc} F_0(\mathcal{H}_S) & F_1(\mathcal{H}_S) & F_{01}(\mathcal{H}_S) & F_2^x(\mathcal{H}_S) \\ P_{\widetilde{S}:0} & P_{\widetilde{S}:1} & P_{\widetilde{S}:01} & P_{\widetilde{S}:2} \\ \{|\widetilde{0}_S\rangle\} & \{|\widetilde{S}:1,n\rangle\}_n & \{|\widetilde{0}_S\rangle\} \cup \{|\widetilde{S}:1,n\rangle\}_n & \{|\widetilde{S}:2,m\rangle\}_m \end{array}$$

We use an analogous notation for the second quantizations of the target space. Second quantizations of the subspaces $\mathcal{H}_{s1}$ is also used, with the following notation:

$$\begin{array}{cccc} F_0(\mathcal{H}_{s1}) & F_1(\mathcal{H}_{s1}) & F_{01}(\mathcal{H}_{s1}) & F_2^x(\mathcal{H}_{s1}) \\ P_{\widetilde{s1}:0} & P_{\widetilde{s1}:1} & P_{\widetilde{s1}:01} & P_{\widetilde{s1}:2} \\ \{|\widetilde{0}_{s1}\rangle\} & \{|\widetilde{s1},k\rangle\}_k & \{|\widetilde{0}_{s1}\rangle\} \cup \{|\widetilde{s1},k\rangle\}_k & \{|\widetilde{s1}:2,m\rangle\}_m \end{array}$$

Similarly for $\mathcal{H}_{s2}$, $\mathcal{H}_{t1}$, $\mathcal{H}_{t2}$. To make more clear when an object (vector, operator, CPM) 'belongs' to a second quantized space it carries a tilde, while objects belonging to a first quantized space do not.

Restriction in source space and *restriction in target space* of CPMs have been defined [2]. Given a subspace $\mathcal{H}_{s1}$ in the source space of a CPM, the *restriction in source space* of ϕ, to $\mathcal{H}_{s1}$ (or just *restriction in source*), is defined as the restriction of ϕ to the subspace $\mathcal{L}(\mathcal{H}_{s1})$. Given a subspace $\mathcal{H}_{t1}$ of the target space of a CPM ϕ the *restriction in target space* is defined as $\chi(Q) = P_{t1}\phi(Q)P_{t1}$ for all $Q \in \mathcal{L}(\mathcal{H}_S)$. It is to be noted that there is a slight abuse of notation in this expression. χ is intended to have $\mathcal{H}_{t1}$ as its target space and not $\mathcal{H}_T$. The facts to be used here is that the restriction in source or target of a CPM, is a CPM. Moreover, the restriction in source space of a trace preserving CPM is trace preserving [2].

Given a CPM $\widetilde{\phi}$ with source $F^x(\mathcal{H}_S)$ and target $F^{x'}(\mathcal{H}_T)$, define the *1-restriction* of $\widetilde{\phi}$ to be the CPM which is the restriction in source to $F_1(\mathcal{H}_S)$ and restriction in target to $F_1(\mathcal{H}_T)$, of $\widetilde{\phi}$. To handle 1-restrictions it will prove convenient to use the following operators:

$$
\begin{aligned}
M_S &= \sum_k |\widetilde{s1,k}\rangle|\widetilde{0}_{s2}\rangle\langle s1,k| + \sum_l |\widetilde{0}_{s1}\rangle|\widetilde{s2,l}\rangle\langle s2,l|,\\
M_T &= \sum_n |\widetilde{t1,n}\rangle|\widetilde{0}_{t2}\rangle\langle t1,n| + \sum_m |\widetilde{0}_{t1}\rangle|\widetilde{t2,m}\rangle\langle t2,m|,\\
M_{s1} &= \sum_k |\widetilde{s1,k}\rangle\langle s1,k|, \quad M_{s2} = \sum_l |\widetilde{s2,l}\rangle\langle s2,l|,\\
M_{t1} &= \sum_n |\widetilde{t1,n}\rangle\langle t1,n|, \quad M_{t2} = \sum_m |\widetilde{t2,m}\rangle\langle t2,m|.
\end{aligned}
$$

In terms of these operators the 1-restriction of a CPM $\widetilde{\phi}$ can be expressed as

$$\phi(Q) = M_T^\dagger\widetilde{\phi}(M_S Q M_S^\dagger)M_T. \tag{1}$$

An operator $V : \mathcal{H}_{s1} \to \mathcal{H}_{t1}$ will in some expressions be treated as a mapping from $\mathcal{H}_S$ to $\mathcal{H}_T$. In such cases it is implicitly assumed that V acts as the zero operator on $\mathcal{H}_{s2}$, and that V is linearly extended to $\mathcal{H}_S$. The range of the original V is some subspace of $\mathcal{H}_{t1}$, but we instead regard it as a subspace of $\mathcal{H}_T$. This convention is for example used in equation (3) in proposition 2. Another notational simplification, in the same spirit as the previous one, concerns CPMs. Given a CPM ϕ with source $\mathcal{H}_{s1}$ and target $\mathcal{H}_{t1}$, it will in some cases be treated as if it had source space $\mathcal{H}_S$ and target $\mathcal{H}_T$. Given a Kraus representation $\{V_k\}_k$ of ϕ we re-interpret V_k as being mappings from $\mathcal{H}_S$ to $\mathcal{H}_T$, as mentioned above. This new set of operators forms a Kraus representation of a CPM with source space $\mathcal{H}_S$ and target space $\mathcal{H}_T$. A typical example of this is $\phi = \phi_1 + \phi_2$, where ϕ_1 has source $\mathcal{H}_{s1}$ and target $\mathcal{H}_{t1}$, and ϕ_2 has source $\mathcal{H}_{s2}$ and target $\mathcal{H}_{t2}$.

A final remark concerns linear maps with $\mathcal{L}(\mathcal{H})$ as its domain of definition. If a linear map Λ fulfils $\Lambda(|\psi\rangle\langle\psi|) = 0$ for all $|\psi\rangle \in \mathcal{H}_{s1}$, then this statement is equivalent to: $\Lambda(\rho) = 0$ for all density operators ρ on $\mathcal{H}_{s1}$, and is moreover equivalent to: $\Lambda(Q) = 0$, $\forall Q \in \mathcal{L}(\mathcal{H}_{s1})$. This is the case, since every density operator can be written as a convex combination of outer products of elements in $\mathcal{H}$, and since every $Q \in \mathcal{L}(\mathcal{H}_{s1})$ can be written as a (complex) linear combination of four density operators. We will in the following pass between these equivalent formulations without any comment.

3. Definition of subspace locality

We begin with a brief description of some of the main ideas appearing in this section. The key observation is the following: $F^x(\mathcal{H}_1 \oplus \mathcal{H}_2) = F^x(\mathcal{H}_1) \otimes F^x(\mathcal{H}_2)$. Hence, when passing to the second quantization in the occupation number representation, the original orthogonal sum gives rise to a tensor product. With respect to this tensor product the subspace local channels can be defined as those which correspond to product channels on these product spaces.

To use channels on the second quantized spaces to represent channels on the first quantized spaces, introduce some problems. When passing from the first quantized description to a second quantized description, we allow more operations, as one may change the total particle number in the system. If an operation on the second quantized space is to act as a channel on the first quantized space, we must make sure that initial single-particle states is mapped to single-particle states. The channels which *respects 1-states* are those for which there corresponds a channel on the first quantized spaces. The second problem is that to a specific channel on first quantized spaces corresponds several possible channels on the second quantized spaces. The *1-restriction* singles out those channels which correspond to a specific channel on the first quantized spaces. A third problem is that the type of second quantization is not unique. Although one may argue, from an intuitive point of view, that the type of second quantization should not matter, since we are dealing with single particles, this has to be proved. Proposition 1 shows that the type of second quantization indeed does not matter. It also shows that, for the questions dealt with here, only parts of the second quantized spaces are relevant and that the definition of subspace locality can be reformulated in terms of these subspaces.

Definition 1 Let the CPM $\widetilde{\phi}$ have source $F^x(\mathcal{H}_S)$ and target $F^{x'}(\mathcal{H}_T)$. We say that $\widetilde{\phi}$ *respects 1-states* if $\mathrm{Tr}(P^{\perp}_{\widetilde{T}:1}\widetilde{\phi}(|\widetilde{\psi}\rangle\langle\widetilde{\psi}|)) = 0$, $\forall|\widetilde{\psi}\rangle \in F_1(\mathcal{H}_S)$, where $P^{\perp}_{\widetilde{T}:1}$ denotes the projector onto the orthogonal complement of $F_1(\mathcal{H}_T)$. Likewise $\widetilde{\phi}$ *respects 0-states* if $\mathrm{Tr}(P^{\perp}_{\widetilde{T}:0}\widetilde{\phi}(|\widetilde{0}_S\rangle\langle\widetilde{0}_S|)) = 0$, where $P^{\perp}_{\widetilde{T}:0}$ is the projector onto the orthogonal complement of $F_0(\mathcal{H}_T)$. If $\widetilde{\phi}$ both respects 1-states and 0-states we say that $\widetilde{\phi}$ *respects 1,0-states*.

In words, a CPM on a second quantized space respects 1-states if no single-particle state is mapped outside the set of strict single-particle states. (We make no restriction on how $\widetilde{\phi}$ may act on states outside the strict single-particle states.) It is to be noted that these definitions will be used also for CPMs with source $F_{01}(\mathcal{H}_S)$ and target $F_{01}(\mathcal{H}_T)$.

Lemma 1 *Let $\widetilde{\phi}$ be a CPM with source $F^x(\mathcal{H}_S)$ and target $F^{x'}(\mathcal{H}_T)$, or alternatively with source $F_{01}(\mathcal{H}_S)$ and target $F_{01}(\mathcal{H}_T)$. If $\widetilde{\phi}$ respects n-states ($n = 0$ or 1), then $P_{\widetilde{T}:n}\widetilde{\phi}(\widetilde{Q})P_{\widetilde{T}:n} = \widetilde{\phi}(\widetilde{Q})$ for all $\widetilde{Q} \in \mathcal{L}(F_n(\mathcal{H}_S))$.*

proof. Let $\{\widetilde{V}_k\}_k$ be some Kraus representation of $\widetilde{\phi}$. If $\widetilde{\phi}$ respects n-states it follows by definition that $\sum_k \mathrm{Tr}(P^{\perp}_{\widetilde{T}:n}\widetilde{V}_k|\widetilde{\psi}\rangle\langle\widetilde{\psi}|\widetilde{V}_k^{\dagger}) = 0$, $\forall|\widetilde{\psi}\rangle \in F_n(\mathcal{H}_S)$. It follows that $\langle\widetilde{\chi}|\widetilde{V}_k|\widetilde{\psi}\rangle = 0$ for all k, all $|\widetilde{\chi}\rangle \in F_n(\mathcal{H}_T)^{\perp}$, and all $|\widetilde{\psi}\rangle \in F_n(\mathcal{H}_S)$. Hence $P^{\perp}_{\widetilde{T}:n}\widetilde{V}_k P_{\widetilde{S}:n} = 0$, from which the lemma follows. □

To single out those channels on the second quantized systems which correspond to channels on the first quantized system, the following lemma is useful, since it gives us the choice to either show trace preservation of the 1-restriction, or to show that the channel on the second quantized system respects 1-states.

Lemma 2 *Let $\widetilde{\Phi}$ be a trace preserving CPM with source $F^{x}(\mathcal{H}_S)$ and target $F^{x'}(\mathcal{H}_T)$, or alternatively with source $F_{01}(\mathcal{H}_S)$ and target $F_{01}(\mathcal{H}_T)$. Then $\widetilde{\Phi}$ respects 1-states if and only if the 1-restriction is trace preserving.*

proof. By assuming that the 1-restriction is trace preserving, it follows that $\mathrm{Tr}(M_T^{\dagger}\widetilde{\Phi}(M_S|\psi\rangle\langle\psi|M_S^{\dagger})M_T) = 1$, for all normalized $|\psi\rangle \in \mathcal{H}_S$. By combining $M_T M_T^{\dagger} = P_{\widetilde{T}:1}$, $P_{\widetilde{T}:1} + P_{\widetilde{T}:1}^{\perp} = \hat{1}$ and the assumption that $\widetilde{\Phi}$ is trace preserving, follows $\mathrm{Tr}(P_{\widetilde{T}:1}^{\perp}\widetilde{\Phi}(M_S|\psi\rangle\langle\psi|M_S^{\dagger})) = 0$. Since M_S is a bijection between $\mathcal{H}_S$ and $F_1(\mathcal{H}_S)$, it follows from the last expression that $\widetilde{\Phi}$ respects 1-states. Reversing the argument above gives that if $\widetilde{\Phi}$ respects 1-states, then the 1-restriction is trace preserving. □

We are now in position to state the definition of subspace locality of channels.

Definition 2 Let Φ be a trace preserving CPM with source $\mathcal{H}_{s1} \oplus \mathcal{H}_{s2}$ and target $\mathcal{H}_{t1} \oplus \mathcal{H}_{t2}$. If there exist trace preserving CPMs $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$, where $\widetilde{\Phi}_1$ has source $F^{x}(\mathcal{H}_{s1})$ and target $F^{x'}(\mathcal{H}_{t1})$, and where $\widetilde{\Phi}_2$ has source $F^{x}(\mathcal{H}_{s2})$ and target $F^{x'}(\mathcal{H}_{t2})$, for some choice of x, x', such that $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$ has Φ as 1-restriction, then we say that Φ is *subspace local* from $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$ to $(\mathcal{H}_{t1}, \mathcal{H}_{t2})$.

In words this definition says that a CPM Φ is subspace local from $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$ to $(\mathcal{H}_{t1}, \mathcal{H}_{t2})$ if there exist some second quantizations of the source space and target space, and some product channel $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$, such that $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$ 'acts' like Φ on the single-particle states. Hence we use the familiar definition of 'local' channel, to define what should be meant by subspace local channel.

It will now be shown that a large part of the second quantized space is irrelevant to the analysis of SL channels. A consequence is that the set of subspace local channels is independent of the choice of statistics in the second quantization.

Let $|\widetilde{\psi}_1\rangle$ be an arbitrary normalized state in $F_{01}(\mathcal{H}_{t1})$. Consider the CPM $\widetilde{\Theta}_1$, with source and target $F^{x'}(\mathcal{H}_{t1})$, defined by

$$\begin{aligned}\widetilde{\Theta}_1(\widetilde{Q}_1) &= P_{\widetilde{t1}:01}\widetilde{Q}_1 P_{\widetilde{t1}:01} + \sum_{l} |\widetilde{\psi}_1\rangle\langle\widetilde{t1}:2,l|\widetilde{Q}_1|\widetilde{t1}:2,l\rangle\langle\widetilde{\psi}_1| \\ &= P_{\widetilde{t1}:01}\widetilde{Q}_1 P_{\widetilde{t1}:01} + |\widetilde{\psi}_1\rangle\langle\widetilde{\psi}_1|\,\mathrm{Tr}(P_{\widetilde{t1}:2}\widetilde{Q}_1), \quad \forall \widetilde{Q}_1 \in \mathcal{L}(F^{x'}(\mathcal{H}_{t1})). \end{aligned} \tag{2}$$

In case $F_2^{x'}(\mathcal{H}_{t1})$ is zero-dimensional, discard the second term in (2). By the form of $\widetilde{\Theta}_1$ it follows that it is a CPM and moreover a trace preserving CPM. Construct analogously a trace preserving CPM $\widetilde{\Theta}_2$, with source and target space $F^{x'}(\mathcal{H}_{t2})$, using a normalized state $|\widetilde{\psi}_2\rangle \in F_{01}(\mathcal{H}_{t2})$.

Lemma 3 *Let $\widetilde{\phi} = \widetilde{\phi}_1 \otimes \widetilde{\phi}_2$, where the CPM $\widetilde{\phi}_1$ has source $F^{x}(\mathcal{H}_{s1})$ and target $F^{x'}(\mathcal{H}_{t1})$, and where the CPM $\widetilde{\phi}_2$ has source $F^{x}(\mathcal{H}_{s2})$ and target $F^{x'}(\mathcal{H}_{t2})$. Let $\widetilde{\phi}' = \widetilde{\phi}_1' \otimes \widetilde{\phi}_2'$ be defined by $\widetilde{\phi}_1' = \widetilde{\Theta}_1 \circ \widetilde{\phi}_1$, $\widetilde{\phi}_2' = \widetilde{\Theta}_2 \circ \widetilde{\phi}_2$.*

- *If $\widetilde{\phi}$ respects 1-states then so does $\widetilde{\phi}'$.*
- *If $\widetilde{\phi}$ respects 1-states, then $\widetilde{\phi}$ and $\widetilde{\phi}'$ have the same 1-restriction.*
- *If $\widetilde{\phi}$ is trace preserving, then so is $\widetilde{\phi}'$.*

proof. We note the following: $(\langle\widetilde{t1}:2,l|\langle\widetilde{t2}:2,l'|)P_{\widetilde{T}:1} = 0$ for every l, l', $(P_{\widetilde{t1}:01} \otimes |\widetilde{\psi}_2\rangle\langle\widetilde{t2}:2,l'|)P_{\widetilde{T}:1} = 0$ for every l', $(|\widetilde{\psi}_1\rangle\langle\widetilde{t1}:2,l| \otimes P_{\widetilde{t2}:01})P_{\widetilde{T}:1} = 0$ for every l, while $(P_{\widetilde{t1}:01} \otimes P_{\widetilde{t2}:01})P_{\widetilde{T}:1} = P_{\widetilde{T}:1}$. Hence follows $[\widetilde{\Theta}_1 \otimes \widetilde{\Theta}_2](P_{\widetilde{T}:1}\widetilde{Q}P_{\widetilde{T}:1}) = P_{\widetilde{T}:1}\widetilde{Q}P_{\widetilde{T}:1}$.

By combining the last expression with lemma 1 and the definition of respect of 1-states, the first part of the lemma follows.

For the second part, note that $M_S|\psi\rangle \in F_1(\mathcal{H}_S)$, for any $|\psi\rangle \in \mathcal{H}_S$. By lemma 1 follows $M_T^\dagger \widetilde{\phi}'(M_S|\psi\rangle\langle\psi|M_S^\dagger)M_T = M_T^\dagger[\widetilde{\Theta}_1 \otimes \widetilde{\Theta}_2]\left(P_{\widetilde{T}:1}\phi(M_S|\psi\rangle\langle\psi|M_S^\dagger)P_{\widetilde{T}:1}\right) = M_T^\dagger\widetilde{\phi}(M_S|\psi\rangle\langle\psi|M_S^\dagger)$, which proves the second part of the lemma.

For the third par we note that both $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_2$ are trace preserving, hence $\widetilde{\Theta}_1\otimes\widetilde{\Theta}_2$ is trace preserving. From this follows that $\widetilde{\phi}'$ is trace preserving if $\widetilde{\phi}$ is. □

Proposition 1 *Let Φ be a trace preserving CPM with source $\mathcal{H}_{s1} \oplus \mathcal{H}_{s2}$ and target $\mathcal{H}_{t1} \oplus \mathcal{H}_{t2}$. Φ is SL from $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$ to $(\mathcal{H}_{t1}, \mathcal{H}_{t2})$, if and only if there exists a trace preserving CPM $\widetilde{\Psi} = \widetilde{\Psi}_1 \otimes \widetilde{\Psi}_2$, where $\widetilde{\Psi}_1$ has source $F_{01}(\mathcal{H}_{s1})$ and target $F_{01}(\mathcal{H}_{t1})$, and where $\widetilde{\Psi}_2$ has source $F_{01}(\mathcal{H}_{s2})$ and target $F_{01}(\mathcal{H}_{t2})$, such that $\widetilde{\Psi}$ has Φ as 1-restriction.*

proof. We begin with the “only if” part of the proposition. Suppose Φ is subspace local from $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$ to $(\mathcal{H}_{t1}, \mathcal{H}_{t2})$. By definition there exists a trace preserving product CPM $\widetilde{\Phi} = \widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$, which has Φ as its 1-restriction. If the construction in lemma 3 is applied, a product CPM $\widetilde{\Phi}' = \widetilde{\Phi}_1' \otimes \widetilde{\Phi}_2'$ is obtained. Lemma 3 gives that $\widetilde{\Phi}'$ is trace preserving and has Φ as its 1-restriction. Let $\widetilde{\Psi}_1$ be the restriction in source to $F_{01}(\mathcal{H}_{s1})$, and in target to $F_{01}(\mathcal{H}_{t1})$, of $\widetilde{\Phi}_1'$. Let $\widetilde{\Psi}_2$ be the restriction in source to $F_{01}(\mathcal{H}_{s2})$, and in target to $F_{01}(\mathcal{H}_{t2})$, of $\widetilde{\Phi}_2'$. Since $F_1(\mathcal{H}_S) \subset F_{01}(\mathcal{H}_{s1}) \otimes F_{01}(\mathcal{H}_{s2})$ and $F_1(\mathcal{H}_T) \subset F_{01}(\mathcal{H}_{t1}) \otimes F_{01}(\mathcal{H}_{t2})$, it follows that the 1-restriction of $\widetilde{\Psi} = \widetilde{\Psi}_1 \otimes \widetilde{\Psi}_2$ is the same as the 1-restriction of $\widetilde{\Phi}'$. Finally it follows from the construction of $\widetilde{\Phi}_1'$ and $\widetilde{\Phi}_2'$, that $\widetilde{\Psi}_1$ and $\widetilde{\Psi}_2$ are trace preserving.

We turn to the “if-part”. $\widetilde{\Psi}_1$ and $\widetilde{\Psi}_2$ are trace preserving CPMs such that $\widetilde{\Psi} = \widetilde{\Psi}_1 \otimes \widetilde{\Psi}_2$ has Φ as 1-restriction. Assume $F_2^x(\mathcal{H}_{s1})$ is not zero-dimensional and let the CPM $\widetilde{\eta}_1$ be defined by $\widetilde{\eta}_1(\widetilde{Q}_1) = \sum_l |\widetilde{0}_{t1}\rangle\langle \widetilde{s1}, 2 : l|\widetilde{Q}_1|\widetilde{s1}, 2 : l\rangle\langle\widetilde{0}_{t1}|$, $\forall \widetilde{Q}_1 \in \mathcal{L}(F^x(\mathcal{H}_{s1}))$. Define $\widetilde{\Phi}_1 = \widetilde{\Psi}_1 + \widetilde{\eta}_1$. By construction it follows that $\widetilde{\Phi}_1$ is a trace preserving CPM. If $F_2^x(\mathcal{H}_{s1})$ is zero-dimensional, the term $\widetilde{\eta}_1$ drops out. In analogy with $\widetilde{\eta}_1$, a CPM $\widetilde{\eta}_2$ can be constructed, from which we define $\widetilde{\Phi}_2 = \widetilde{\Psi}_2 + \widetilde{\eta}_2$. If $\widetilde{\Psi}_1 \otimes \widetilde{\Psi}_2$ has Φ as 1-restriction, so does $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$. Hence Φ is subspace local. □

4. Local subspace preserving channels

Here a special class of SL channels is defined, the *local subspace preserving* (LSP) channels. Intuitively these are the channels which are subspace local and which do not transport any probability weight from one location to the other. It is shown in section 5 that the set of LSP channels is the intersection between the set of subspace preserving channels (SP) [1] and the set of SL channels.

Definition 3 Let Φ be a trace preserving CPM with source $\mathcal{H}_{s1} \oplus \mathcal{H}_{s2}$ and target $\mathcal{H}_{t1} \oplus \mathcal{H}_{t2}$. We say that Φ is *local subspace preserving* (LSP) from $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$ to $(\mathcal{H}_{t1}, \mathcal{H}_{t2})$ if there exists a trace preserving CPM $\widetilde{\Phi}_1$ with source $F_{01}(\mathcal{H}_{s1})$ and target $F_{01}(\mathcal{H}_{t1})$, and a trace preserving CPM $\widetilde{\Phi}_2$ with source $F_{01}(\mathcal{H}_{s2})$ and target $F_{01}(\mathcal{H}_{t2})$, such that $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$ has Φ as 1-restriction and such that both $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ respects 1,0-states.

The following proposition gives an explicit description of the set of LSP channels. It does so without referring to any second quantized space. In other words, the definition of LSP channels as formulated in the second quantized spaces, is ‘brought back’ to the first quantized spaces.

Proposition 2 *A trace preserving CPM Φ is LSP from $(\mathcal{H}_{s1},\mathcal{H}_{s2})$ to $(\mathcal{H}_{t1},\mathcal{H}_{t2})$ if and only if there exists a linearly independent Kraus representation $\{V_n\}_{n=1}^{N}\subset\mathcal{L}(\mathcal{H}_{t1},\mathcal{H}_{s1})$ of some trace preserving CPM with source $\mathcal{H}_{s1}$ and target $\mathcal{H}_{t1}$, and a linearly independent Kraus representation $\{W_m\}_{m=1}^{M}\subset\mathcal{L}(\mathcal{H}_{t2},\mathcal{H}_{s2})$ of some trace preserving CPM with source $\mathcal{H}_{s2}$ and target $\mathcal{H}_{t2}$, such that*

$$\Phi(Q)=\sum_{n=1}^{N}V_nQV_n^{\dagger}+\sum_{m=1}^{M}W_mQW_m^{\dagger}+VQW^{\dagger}+WQV^{\dagger},\quad \forall Q\in\mathcal{L}(\mathcal{H}_S),\tag{3}$$

$$V=\sum_{n=1}^{N}c_{1,n}V_n,\quad W=\sum_{m=1}^{M}c_{2,m}W_m,\tag{4}$$

where the vectors $c_1=[c_{1,n}]_{n=1}^{N}$ and $c_2=[c_{1,m}]_{m=1}^{M}$ fulfil the conditions

$$||c_1||^2=\sum_{n=1}^{N}|c_{1,n}|^2\leq 1,\quad ||c_2||^2=\sum_{m=1}^{M}|c_{2,m}|^2\leq 1.\tag{5}$$

Note that this proposition is stated in terms of two linearly independent Kraus representations. There exists another equivalent formulation of this proposition in terms of arbitrary bases of $\mathcal{L}(\mathcal{H}_{s1},\mathcal{H}_{t1})$ and $\mathcal{L}(\mathcal{H}_{s2},\mathcal{H}_{t2})$. This is analogous to the relation between proposition 10 in [1] and proposition 3 in [2]. The alternative formulation of proposition 2 can be obtained from proposition 10 in [1] by adding the condition that the matrix C should be possible to write as $C=c_1c_2^{\dagger}$, for some vector $c_1\in\mathbb{C}^K$ and some $c_2\in\mathbb{C}^L$. Moreover, a condition for trace preservation must be added, which takes the form $\sum_{kk'}a_{k,k'}V_{k'}^{\dagger}V_k=P_{s1}$ and $\sum_{ll'}a_{l,l'}W_{l'}^{\dagger}W_l=P_{s2}$. (Here the notation in proposition 10 in [1] has been used.)

proof. By definition, Φ is an LSP channel if and only if there exist trace preserving CPMs $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$, such that $\widetilde{\Phi}_1\otimes\widetilde{\Phi}_2$ has Φ as 1-restriction, and such that both $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ respects 1,0-states. Since $\widetilde{\Phi}_1$ has source $F_{01}(\mathcal{H}_{s1})$ and target $F_{01}(\mathcal{H}_{t1})$, the condition that $\widetilde{\Phi}_1$ respects 1,0-states is equivalent to $\widetilde{\Phi}_1$ being SP from $(F_0(\mathcal{H}_{s1}),F_1(\mathcal{H}_{s1}))$ to $(F_0(\mathcal{H}_{s2}),F_1(\mathcal{H}_{s2}))$ (compare with definition of SP in [1]). According to proposition 1 in [2], $\widetilde{\Phi}_1$ is SP if and only if it is a trace preserving gluing of two trace preserving CPMs. One of these having source $F_0(\mathcal{H}_{s1})$ and target $F_0(\mathcal{H}_{t1})$, and the other having source $F_1(\mathcal{H}_{s1})$ and target $F_1(\mathcal{H}_{t1})$. The spaces $F_0(\mathcal{H}_{s1})$ and $F_0(\mathcal{H}_{t1})$ are one-dimensional and there exists only one trace preserving CPM with these as source and target spaces, namely the CPM with linearly independent Kraus representation $\{|\widetilde{0}_{t1}\rangle\langle\widetilde{0}_{s1}|\}$. The other trace preserving CPM, with source $F_1(\mathcal{H}_{s1})$ and target $F_1(\mathcal{H}_{t1})$, has some linearly independent Kraus representation $\{\widetilde{V}_n\}_{n=1}^{N}$. According to proposition 3 in [2], $\widetilde{\Phi}_1$ is an SP channel, if and only if it can be written as

$$\begin{aligned}\widetilde{\Phi}_1(\widetilde{Q}_1)=&\sum_n\widetilde{V}_n\widetilde{Q}_1\widetilde{V}_n^{\dagger}+|\widetilde{0}_{t1}\rangle\langle\widetilde{0}_{s1}|\widetilde{Q}_1|\widetilde{0}_{s1}\rangle\langle\widetilde{0}_{t1}|\\&+\sum_n c_{1,n}\widetilde{V}_n\widetilde{Q}_1|\widetilde{0}_{s1}\rangle\langle\widetilde{0}_{t1}|+\sum_n c_{1,n}^{*}|\widetilde{0}_{t1}\rangle\langle\widetilde{0}_{s1}|\widetilde{Q}_1\widetilde{V}_n^{\dagger},\end{aligned}\tag{6}$$

for all $\widetilde{Q}_1\in\mathcal{L}(F_{01}(\mathcal{H}_{s1}))$, where $c_1=[c_{1,n}]_{n=1}^{N}$ fulfils the condition $||c_1||^2=\sum_{n=1}^{N}|c_{1,n}|^2\leq 1$. By an analogous reasoning, $\widetilde{\Phi}_2$ is an SP channel if and only if

$$\widetilde{\Phi}_2(\widetilde{Q}_2)=\sum_m\widetilde{W}_m\widetilde{Q}_2\widetilde{W}_m^{\dagger}+|\widetilde{0}_{t2}\rangle\langle\widetilde{0}_{s2}|\widetilde{Q}_2|\widetilde{0}_{s2}\rangle\langle\widetilde{0}_{t2}|$$

$$+\sum_m c_{2,m}\widetilde{W}_m\widetilde{Q}_2|\widetilde{0}_{s2}\rangle\langle\widetilde{0}_{t2}| + \sum_m c_{2,m}^*|\widetilde{0}_{t2}\rangle\langle\widetilde{0}_{s2}|\widetilde{Q}_2\widetilde{W}_m^\dagger, \quad (7)$$

for all $\widetilde{Q}_2 \in \mathcal{L}(F_{01}(\mathcal{H}_{s2}))$, where $c_2 = [c_{2,m}]_{m=1}^M$ fulfils the condition $||c_2||^2 = \sum_{m=1}^M |c_{2,m}|^2 \leq 1$, and where $\{\widetilde{W}_m\}_{m=1}^M$ is a linearly independent Kraus representation of some channel with source $F_1(\mathcal{H}_{s2})$ and target $F_1(\mathcal{H}_{t2})$.

The "only if" part of the proposition is now possible to prove by taking the 1-restriction of $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$ using (1). Doing so one obtains expression (3), with $V_n = M_{t1}^\dagger \widetilde{V}_n M_{s1}$ and $W_m = M_{t2}^\dagger \widetilde{W}_m M_{s2}$. It is possible to check that $\{V_n\}_{n=1}^N$ is a linearly independent Kraus representation [1] of a trace preserving CPM with source $\mathcal{H}_{s1}$ and target $\mathcal{H}_{t1}$. (One can make use of $M_{s1}M_{s1}^\dagger = P_{\widetilde{s}1:1}$, $M_{t1}M_{t1}^\dagger = P_{\widetilde{t}1:1}$, and $P_{\widetilde{t}1:1}\widetilde{V}_n P_{\widetilde{s}1:1} = \widetilde{V}_n$.) Analogously one can show that $\{W_m\}_{m=1}^M$ is a linearly independent Kraus representation of a channel with source $\mathcal{H}_{s2}$ and target $\mathcal{H}_{t2}$.

For the "if" part, define $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ via (6) and (7), with $\widetilde{V}_n = M_{t1}V_nM_{s1}^\dagger$ and $\widetilde{W}_m = M_{t2}W_mM_{s2}^\dagger$. Both $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ respects 1,0-states and $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$ has Φ as 1-restriction. □

4.1. Unitary representation of a subclass of the LSP channels

A special class of LSP channels are those which have identical source and target spaces and moreover where the separation in subspaces in the source and target are identical. $\mathcal{H}_T = \mathcal{H}_S$, $\mathcal{H}_{t1} = \mathcal{H}_{s1}$ and $\mathcal{H}_{t2} = \mathcal{H}_{s2}$. An example is a particle in a two-path interferometer. To ease terminology a bit we say that a CPM which is SP, LSP, or SL from $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$ to $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$, is SP, LSP, or SL *on* $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$.

Here the channels in this subclass are constructed as a joint unitary evolution of the system and an ancilla system. It is well known [3] that any channel with identical source and target spaces can be constructed via a joint unitary evolution. Here we prove a special form of unitary representation.

Proposition 3 *Φ is LSP on $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$ if and only if there exist finite-dimensional Hilbert spaces $\mathcal{H}_{a1}$ and $\mathcal{H}_{a2}$ and normalized vectors $|a1\rangle \in \mathcal{H}_{a1}$,$|a2\rangle \in \mathcal{H}_{a2}$, an operator V_1 on $\mathcal{H}_S \otimes \mathcal{H}_{a1}$ and V_2 on $\mathcal{H}_S \otimes \mathcal{H}_{a2}$, such that*

$$V_1V_1^\dagger = V_1^\dagger V_1 = P_{s1} \otimes \hat{1}_{a1}, \quad V_2V_2^\dagger = V_2^\dagger V_2 = P_{s2} \otimes \hat{1}_{a2}, \quad (8)$$

and such that Φ can be written

$$\Phi(Q) = \mathrm{Tr}_{a1,a2}\left(UQ \otimes |a1\rangle\langle a1| \otimes |a2\rangle\langle a2|U^\dagger\right), \quad \forall Q \in \mathcal{L}(\mathcal{H}_S), \quad (9)$$

where $U = V_1 \otimes \hat{1}_{a2} + V_2 \otimes \hat{1}_{a1}$ is a unitary operator.

To have some intuition about this representation, note that the unitary operator U is decomposed into two parts. One part which acts non-trivially only on the subspace $\mathcal{H}_{s1}$ and ancilla $a1$, and another part which only acts on the subspace $\mathcal{H}_{s2}$ and the second ancilla $a2$. Initially the two ancilla systems are in a product state. Hence we have a 'local resource' in form of two uncorrelated ancilla systems, one on each location. In some sense, the evolution is such that it only acts 'locally', since if the state of the system is in subspace $\mathcal{H}_{s1}$, it only interacts with ancilla system $a1$, and if in subspace $\mathcal{H}_{s2}$, it only interacts with ancilla $a2$. This seem to give some intuitive support to the here suggested definition of subspace locality as being a reasonable definition. A more clear cut example is given in [4], where the expression (9) is applied to the special case of a two-path single particle interferometer of particles with internal degree of freedom.

We may compare proposition 3 with proposition 11 in [1]. There it is shown that, under the same restrictions on the source and target spaces, there is a unitary representation for SP channels. The expression derived there is similar to the expression in proposition 3. The difference is that for the SP channels there is only *one* ancilla system and not two as in the case of LSP channels. In the case of SP channels, both the 'locations' interact with the same ancilla system, which perhaps give some support to the idea that a general SP channel should not be regarded as being 'subspace local'.

proof. Suppose Φ is LSP, then there exists a product channel of trace preserving CPMs $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$, with Φ as 1-restriction. Consider $\widetilde{\Phi}_1$, which has source $F_{01}(\mathcal{H}_{s1})$ and target $F_{01}(\mathcal{H}_{s1})$. Since Φ is LSP it follows that $\widetilde{\Phi}_1$ is trace preserving and SP on $(F_0(\mathcal{H}_{s1}), F_1(\mathcal{H}_{s1}))$. By proposition 11 in [1], there exists a finite-dimensional Hilbert space $\mathcal{H}_{a1}$, a normalized $|a1'\rangle \in \mathcal{H}_{a1}$, and operators $\widetilde{W}_{1,0}$ and $\widetilde{W}_{1,1}$ on $F_{01}(\mathcal{H}_{s1}) \otimes \mathcal{H}_{a1}$, such that

$$\widetilde{W}_{1,0}\widetilde{W}_{1,0}^{\dagger} = \widetilde{W}_{1,0}^{\dagger}\widetilde{W}_{1,0} = |\widetilde{0}_{s1}\rangle\langle\widetilde{0}_{s1}| \otimes \hat{1}_{a1}, \tag{10}$$

$$\widetilde{W}_{1,1}\widetilde{W}_{1,1}^{\dagger} = \widetilde{W}_{1,1}^{\dagger}\widetilde{W}_{1,1} = P_{\widetilde{s1}:1} \otimes \hat{1}_{a1}, \tag{11}$$

$$\widetilde{\Phi}_1(\widetilde{Q}_1) = \mathrm{Tr}_{a1}(\widetilde{U}_1'\widetilde{Q}_1 \otimes |a1'\rangle\langle a1'|\widetilde{U}_1'^{\dagger}), \quad \forall \widetilde{Q}_1 \in \mathcal{L}(F_{01}(\mathcal{H}_{s1})), \tag{12}$$

where $\widetilde{U}_1' = \widetilde{W}_{1,0} + \widetilde{W}_{1,1}$ is a unitary operator. Since $F_0(\mathcal{H}_{s1})$ is one-dimensional, equation (10) implies $\widetilde{W}_{1,0} = |\widetilde{0}_{s1}\rangle\langle\widetilde{0}_{s1}| \otimes U_{a1}$, where U_{a1} is a unitary operator on $\mathcal{H}_{a1}$. Define $\widetilde{V}_{1,0}$ by

$$\widetilde{V}_{1,0} = \widetilde{W}_{1,0}(\hat{1} \otimes U_{a1}^{\dagger}) = |\widetilde{0}_{s1}\rangle\langle\widetilde{0}_{s1}| \otimes \hat{1}_{a1}.$$

Define $\widetilde{V}_{1,1} = \widetilde{W}_{1,1}(\hat{1} \otimes U_{a1}^{\dagger})$, $|a1\rangle = U_{a1}|a1'\rangle$, and $\widetilde{U}_1 = \widetilde{V}_{1,0} + \widetilde{V}_{1,1}$. Equation (12) still holds with $\widetilde{U}_1'$ changed into $\widetilde{U}_1$ and $|a1'\rangle$ changed into $|a1\rangle$. Moreover, (10) and (11) imply

$$\widetilde{V}_{1,1}\widetilde{V}_{1,1}^{\dagger} = \widetilde{V}_{1,1}^{\dagger}\widetilde{V}_{1,1} = P_{\widetilde{s1}:1} \otimes \hat{1}_{a1}, \quad \widetilde{V}_{2,1}\widetilde{V}_{2,1}^{\dagger} = \widetilde{V}_{2,1}^{\dagger}\widetilde{V}_{2,1} = P_{\widetilde{s2},1} \otimes \hat{1}_{a2}. \tag{13}$$

The operator $\widetilde{U}_1$ is unitary since $\widetilde{U}_1 = \widetilde{U'}_1(\hat{1} \otimes U_{a1})$.

An analogous argument can be applied to $\widetilde{\Phi}_2$, which results in a representation of $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$ as

$$[\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2](\widetilde{Q}) = \mathrm{Tr}_{a1,a2}((\widetilde{U}_1 \otimes \widetilde{U}_2)\widetilde{Q} \otimes |a1\rangle\langle a1| \otimes |a2\rangle\langle a2|(\widetilde{U}_1^{\dagger} \otimes \widetilde{U}_1^{\dagger})), \tag{14}$$

where

$$\widetilde{U}_1 = \widetilde{V}_{1,1} + |\widetilde{0}_{s1}\rangle\langle\widetilde{0}_{s1}| \otimes \hat{1}_{a1}, \quad \widetilde{U}_2 = \widetilde{V}_{2,1} + |\widetilde{0}_{s2}\rangle\langle\widetilde{0}_{s2}| \otimes \hat{1}_{a2}. \tag{15}$$

The next step is to calculate the 1-restriction of the channel (14), using (1) (with $M_T = M_S$). It is useful to note that conditions (13) imply $(P_{\widetilde{s1}:1}\otimes\hat{1}_{a1})\widetilde{V}_{1,1}(P_{\widetilde{s1}:1}\otimes\hat{1}_{a1}) = \widetilde{V}_{1,1}$. This can be proved using a singular value decomposition of $V_{1,1}$. For more details see the proof of proposition 11 in [1]. The result of the 1-restriction is equation (9), with

$$V_1 = (M_{s1}^{\dagger} \otimes \hat{1}_{a1})\widetilde{V}_{1,1}(M_{s1} \otimes \hat{1}_{a1}), \quad V_2 = (M_{s2}^{\dagger} \otimes \hat{1}_{a2})\widetilde{V}_{2,1}(M_{s2} \otimes \hat{1}_{a2}). \tag{16}$$

One can check that the conditions (13) imply (8), and from that showing that U is unitary.

To prove the "if part" of the proposition, define

$$\widetilde{V}_{1,1} = (M_{s1} \otimes \hat{1}_{a1})V_1(M_{s1}^{\dagger} \otimes \hat{1}_{a1}), \quad \widetilde{V}_{2,1} = (M_{s2} \otimes \hat{1}_{a2})V_1(M_{s2}^{\dagger} \otimes \hat{1}_{a2}). \tag{17}$$

Define $\widetilde{U}_1$ and $\widetilde{U}_2$ via (15) and define the CPM $\widetilde{\Phi}_1\otimes\widetilde{\Phi}_2$ via (14). (This is a product CPM by construction). One can check that $\widetilde{\Phi}_1\otimes\widetilde{\Phi}_2$ has Φ as 1-restriction and that $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ each is trace preserving and respects 1,0-states. Hence Φ is an LSP channel. $\square$

5. Subspace local channels

In this section we turn to the SL channels in general and prove explicit expressions to generate all SL channels. It is shown that the set of SL channels decomposes into a union of four disjoint classes of channels, of which the LSP channels forms one such class.

Lemma 4 *Let ϕ be a CPM such that $\mathrm{Tr}(P_{t1}\phi(|\psi_{s1}\rangle\langle\psi_{s1}|)) = 0$, $\forall|\psi_{s1}\rangle\in\mathcal{H}_{s1}$ and $\mathrm{Tr}(P_{t1}\phi(|\psi_{s2}\rangle\langle\psi_{s2}|)) = 0$, $\forall|\psi_{s2}\rangle\in\mathcal{H}_{s2}$. Then $P_{t2}\phi(|\psi\rangle\langle\psi|)P_{t2} = \phi(|\psi\rangle\langle\psi|)$, $\forall|\psi\rangle\in\mathcal{H}_S$.*

proof. Let $\{V_k\}_k$ be an arbitrary Kraus representation of ϕ. According to lemma 1 in [1], the condition $\mathrm{Tr}(P_{t1}\phi(|\psi_{s1}\rangle\langle\psi_{s1}|)) = 0$, $\forall|\psi_{s1}\rangle\in\mathcal{H}_{s1}$ implies $P_{t1}V_kP_{s1} = 0$. The second condition similarly leads to $P_{t1}V_kP_{s2} = 0$. By combining these one obtains $P_{t1}V_k = 0$, which implies $P_{t2}V_k = V_k$, from which the lemma follows. $\square$

Lemma 5 *Let $\widetilde{\Phi}_1$ be a trace preserving CPM with source $F_{01}(\mathcal{H}_{s1})$ and target $F_{01}(\mathcal{H}_{t1})$ and let $\widetilde{\Phi}_2$ be a trace preserving CPM with source $F_{01}(\mathcal{H}_{s2})$ and target $F_{01}(\mathcal{H}_{t2})$. If $\widetilde{\Phi}_1\otimes\widetilde{\Phi}_2$ respects 1-states, then exactly one of the following four cases is true*

- *$\widetilde{\Phi}_1$ is SP from $(F_0(\mathcal{H}_{s1}),F_1(\mathcal{H}_{s1}))$ to $(F_0(\mathcal{H}_{t1}),F_1(\mathcal{H}_{t1}))$ and $\widetilde{\Phi}_2$ is SP from $(F_0(\mathcal{H}_{s2}),F_1(\mathcal{H}_{s2}))$ to $(F_0(\mathcal{H}_{t2}),F_1(\mathcal{H}_{t2}))$.*
- *$\widetilde{\Phi}_1$ is SP from $(F_0(\mathcal{H}_{s1}),F_1(\mathcal{H}_{s1}))$ to $(F_1(\mathcal{H}_{t1}),F_0(\mathcal{H}_{t1}))$ and $\widetilde{\Phi}_2$ is SP from $(F_0(\mathcal{H}_{s2}),F_1(\mathcal{H}_{s2}))$ to $(F_1(\mathcal{H}_{t2}),F_0(\mathcal{H}_{t2}))$.*
- $\widetilde{\Phi}_1(\widetilde{Q}_1) = |\widetilde{0}_{t1}\rangle\langle\widetilde{0}_{t1}|\,\mathrm{Tr}(\widetilde{Q}_1)$, $\quad P_{\tilde{t}2:1}\widetilde{\Phi}_2(\widetilde{Q}_2)P_{\tilde{t}2:1} = \widetilde{\Phi}_2(\widetilde{Q}_2)$,
- $P_{\tilde{t}1:1}\widetilde{\Phi}_1(\widetilde{Q}_1)P_{\tilde{t}1:1} = \widetilde{\Phi}_1(\widetilde{Q}_1)$, $\quad \widetilde{\Phi}_2(\widetilde{Q}_2) = |\widetilde{0}_{t2}\rangle\langle\widetilde{0}_{t2}|\,\mathrm{Tr}(\widetilde{Q}_2)$,

for all $\widetilde{Q}_1\in\mathcal{L}(F_{01}(\mathcal{H}_{s1}))$ and for all $\widetilde{Q}_2\in\mathcal{L}(F_{01}(\mathcal{H}_{s2}))$.

proof. Since $\widetilde{\Phi}_1\otimes\widetilde{\Phi}_2$ respects 1-states, it holds, by definition

$$\begin{aligned}&\mathrm{Tr}\left((P_{\tilde{t}1:0}\otimes P_{\tilde{t}2:0}+P_{\tilde{t}1:1}\otimes P_{\tilde{t}2:1})[\widetilde{\Phi}_1\otimes\widetilde{\Phi}_2](|\widetilde{\psi}\rangle\langle\widetilde{\psi}|)\right) = 0,\\ &\forall|\widetilde{\psi}\rangle\in F_0(\mathcal{H}_{s1})\otimes F_1(\mathcal{H}_{s2})\oplus F_1(\mathcal{H}_{s1})\otimes F_0(\mathcal{H}_{s2}).\end{aligned}\tag{18}$$

Equation (18) implies the following conditions:

$$\begin{aligned}&\mathrm{Tr}(P_{\tilde{t}1:0}\widetilde{\Phi}_1(|\widetilde{\psi}_0^{s1}\rangle\langle\widetilde{\psi}_0^{s1}|))\,\mathrm{Tr}(P_{\tilde{t}2:0}\widetilde{\Phi}_2(|\widetilde{\psi}_1^{s2}\rangle\langle\widetilde{\psi}_1^{s2}|)) = 0,\\ &\quad\forall|\widetilde{\psi}_0^{s1}\rangle\in F_0(\mathcal{H}_{s1}),\quad \forall|\widetilde{\psi}_1^{s2}\rangle\in F_1(\mathcal{H}_{s2}),\end{aligned}\tag{19}$$

$$\begin{aligned}&\mathrm{Tr}(P_{\tilde{t}1:1}\widetilde{\Phi}_1(|\widetilde{\psi}_0^{s1}\rangle\langle\widetilde{\psi}_0^{s1}|))\,\mathrm{Tr}(P_{\tilde{t}2:1}\widetilde{\Phi}_2(|\widetilde{\psi}_1^{s2}\rangle\langle\widetilde{\psi}_1^{s2}|)) = 0,\\ &\quad\forall|\widetilde{\psi}_0^{s1}\rangle\in F_0(\mathcal{H}_{s1}),\quad \forall|\widetilde{\psi}_1^{s2}\rangle\in F_1(\mathcal{H}_{s2}),\end{aligned}\tag{20}$$

$$\begin{aligned}&\mathrm{Tr}(P_{\tilde{t}1:0}\widetilde{\Phi}_1(|\widetilde{\psi}_1^{s1}\rangle\langle\widetilde{\psi}_1^{s1}|))\,\mathrm{Tr}(P_{\tilde{t}2:0}\widetilde{\Phi}_2(|\widetilde{\psi}_0^{s2}\rangle\langle\widetilde{\psi}_0^{s2}|)) = 0,\\ &\quad\forall|\widetilde{\psi}_1^{s1}\rangle\in F_1(\mathcal{H}_{s1}),\quad \forall|\widetilde{\psi}_0^{s2}\rangle\in F_0(\mathcal{H}_{s2}),\end{aligned}\tag{21}$$

$$\mathrm{Tr}(P_{\widetilde{t1}:1}\widetilde{\Phi}_1(|\widetilde{\psi}_1^{s1}\rangle\langle\widetilde{\psi}_1^{s1}|))\,\mathrm{Tr}(P_{\widetilde{t2}:1}\widetilde{\Phi}_2(|\widetilde{\psi}_0^{s2}\rangle\langle\widetilde{\psi}_0^{s2}|)) = 0,$$
$$\forall|\widetilde{\psi}_1^{s1}\rangle \in F_1(\mathcal{H}_{s1}), \quad \forall|\widetilde{\psi}_0^{s2}\rangle \in F_0(\mathcal{H}_{s2}). \tag{22}$$

If condition (19) is to be fulfilled then either

$$\mathrm{Tr}(P_{\widetilde{t1}:0}\widetilde{\Phi}_1(|\widetilde{\psi}_0^{s1}\rangle\langle\widetilde{\psi}_0^{s1}|)) = 0, \quad \forall|\widetilde{\psi}_0^{s1}\rangle \in F_0(\mathcal{H}_{s1}), \tag{23}$$

or

$$\mathrm{Tr}(P_{\widetilde{t2}:0}\widetilde{\Phi}_2(|\widetilde{\psi}_1^{s2}\rangle\langle\widetilde{\psi}_1^{s2}|)) = 0, \quad \forall|\widetilde{\psi}_1^{s2}\rangle \in F_1(\mathcal{H}_{s2}). \tag{24}$$

First suppose (23) is true. By the assumption that $\widetilde{\Phi}_1$ is trace preserving follows

$$\mathrm{Tr}(P_{\widetilde{t1}:1}\widetilde{\Phi}_1(|\widetilde{\psi}_0^{s1}\rangle\langle\widetilde{\psi}_0^{s1}|)) = 1, \quad \forall \text{ normalized } |\widetilde{\psi}_0^{s1}\rangle \in F_0(\mathcal{H}_{s1}). \tag{25}$$

Equation (25) together with condition (20) imply

$$\mathrm{Tr}(P_{\widetilde{t2}:1}\widetilde{\Phi}_2(|\widetilde{\psi}_1^{s2}\rangle\langle\widetilde{\psi}_1^{s2}|)) = 0, \quad \forall|\widetilde{\psi}_1^{s2}\rangle \in F_1(\mathcal{H}_{s2}). \tag{26}$$

Since $\widetilde{\Phi}_2$ is trace preserving it follows from (26) that

$$\mathrm{Tr}(P_{\widetilde{t2}:0}\widetilde{\Phi}_2(|\widetilde{\psi}_1^{s2}\rangle\langle\widetilde{\psi}_1^{s2}|)) = 1, \quad \forall \text{ normalized } |\widetilde{\psi}_1^{s2}\rangle \in F_1(\mathcal{H}_{s2}), \tag{27}$$

which clearly contradicts (24). Hence, if (23) is true then (24) cannot be true. If we on the other hand assume (24) to be true then, by a similar derivation as above, using that $\widetilde{\Phi}_2$ is trace preserving and using (20), we find

$$\mathrm{Tr}(P_{\widetilde{t1}:1}\widetilde{\Phi}_1(|\widetilde{\psi}_0^{s1}\rangle\langle\widetilde{\psi}_0^{s1}|)) = 0, \quad \forall|\widetilde{\psi}_0^{s1}\rangle \in F_0(\mathcal{H}_{s1}). \tag{28}$$

Since $\widetilde{\Phi}_1$ is trace preserving, it is possible to deduce a contradiction to (23) from (28). (Similarly as (27) is in contradiction with (24).) To summarize: either (23) and (26) are true (which we call case a.1), or (24) and (28) are true (to be called case a.2), but not both. By analogous reasoning it is possible to show, using conditions (21) and (22), that either

$$\mathrm{Tr}(P_{\widetilde{t1}:0}\widetilde{\Phi}_1(|\widetilde{\psi}_1^{s1}\rangle\langle\widetilde{\psi}_1^{s1}|)) = 0, \quad \forall|\widetilde{\psi}_1^{s1}\rangle \in F_1(\mathcal{H}_{s1}),$$
$$\mathrm{Tr}(P_{\widetilde{t2}:1}\widetilde{\Phi}_2(|\widetilde{\psi}_0^{s2}\rangle\langle\widetilde{\psi}_0^{s2}|)) = 0, \quad \forall|\widetilde{\psi}_0^{s2}\rangle \in F_0(\mathcal{H}_{s2}), \tag{29}$$

(to be called case b.1) or

$$\mathrm{Tr}(P_{\widetilde{t2}:0}\widetilde{\Phi}_2(|\widetilde{\psi}_0^{s2}\rangle\langle\widetilde{\psi}_0^{s2}|)) = 0, \quad \forall|\widetilde{\psi}_0^{s2}\rangle \in F_0(\mathcal{H}_{s2}),$$
$$\mathrm{Tr}(P_{\widetilde{t1}:1}\widetilde{\Phi}_1(|\widetilde{\psi}_1^{s1}\rangle\langle\widetilde{\psi}_1^{s1}|)) = 0, \quad \forall|\widetilde{\psi}_1^{s1}\rangle \in F_1(\mathcal{H}_{s1}) \tag{30}$$

(to be called called case b.2) is true, but not both. The a-cases are independent of the b-cases. Hence, there is in total four mutually exclusive cases. These four cases are treated separately.

Case (a.1, b.1): According to lemma 4, equation (23) together with the first equation in (29) imply $P_{\widetilde{t1}:1}\widetilde{\Phi}_1(\widetilde{Q}_1)P_{\widetilde{t1}:1} = \widetilde{\Phi}_1(\widetilde{Q}_1)$ for all $\widetilde{Q}_1 \in \mathcal{L}(F_{01}(\mathcal{H}_{s1}))$, which is the first condition in the fourth case of the lemma. Similarly (26) and the second equation in (29) together with lemma 4 imply $P_{\widetilde{t2}:0}\widetilde{\Phi}_2(\widetilde{Q}_2)P_{\widetilde{t2}:0} = \widetilde{\Phi}_2(\widetilde{Q}_2)$, for all $\widetilde{Q}_2 \in \mathcal{L}(F_{01}(H_{s2}))$. Since $F_0(\mathcal{H}_{t2})$ is a one-dimensional space and since $\widetilde{\Phi}_2$ is trace preserving, it follows that $\widetilde{\Phi}_2$ can be written as $\widetilde{\Phi}_2(\widetilde{Q}_2) = |\widetilde{0}_{t2}\rangle\langle\widetilde{0}_{t2}|\,\mathrm{Tr}(\widetilde{Q}_2)$, which is the second condition in the fourth case of the lemma.

Case (a.2, b.2): By very similar calculations as in case (a.1, b.1), it follows that case (a.2, b.2) leads to the third case in the lemma.

Case (a.1, b.2): Equation (23) together with the second equation in (30) imply that $\widetilde{\Phi}_1$ is SP from $(F_0(\mathcal{H}_{s1}), F_1(\mathcal{H}_{s1}))$ to $(F_1(\mathcal{H}_{t1}), F_0(\mathcal{H}_{t1}))$. Likewise (26) together with the first equation in (30) imply that $\widetilde{\Phi}_2$ is SP from $(F_0(\mathcal{H}_{s2}), F_1(\mathcal{H}_{s2}))$ to $(F_1(\mathcal{H}_{t2}), F_0(\mathcal{H}_{t2}))$. Hence, case (a.1, b.2) implies the second case in the lemma.

Case (a.2, b.1): A very similar derivation as in case (a.1, b.2) gives that case (a.2, b.1) implies the first case in the lemma. □

Proposition 4 *If Φ is subspace local from $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$ to $(\mathcal{H}_{t1}, \mathcal{H}_{t2})$, then Φ belongs to exactly one of the following classes*

C_1: *Φ is LSP from $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$ to $(\mathcal{H}_{t1}, \mathcal{H}_{t2})$.*

C_2: *There exists a density operator ρ_1 on $\mathcal{H}_T$ such that $P_{t1}\rho_1 P_{t1} = \rho_1$, with non-zero eigenvalues $\{\lambda_n^1\}_{n=1}^N$ and some corresponding orthonormal set of eigenvectors $\{|\rho_n^1\rangle\}_{n=1}^N$ $(N \leq \dim(\mathcal{H}_{t1}))$. There exists a density operator ρ_2 on $\mathcal{H}_T$ such that $P_{t2}\rho_2 P_{t2} = \rho_2$, with non-zero eigenvalues $\{\lambda_m^2\}_{m=1}^M$ and some corresponding orthonormal set of eigenvectors $\{|\rho_m^2\rangle\}_{m=1}^M$ $(M \leq \dim(\mathcal{H}_{t2}))$. There exists some matrices $C = [C_{n,k}]_{k,n=1}^{K,N}$ and $D = [D_{m,l}]_{m,l=1}^{M,L}$, where $K = \dim(\mathcal{H}_{s1})$, $L = \dim(\mathcal{H}_{s2})$, such that $CC^\dagger \leq I_N$ and $DD^\dagger \leq I_M$, and*

$$\begin{aligned}\Phi(Q) = {} & \rho_1 \operatorname{Tr}(P_{s2}Q) + \rho_2 \operatorname{Tr}(P_{s1}Q) \\ & + \sum_{nm}\sum_{lk} \langle s2,l|Q|s1,k\rangle C_{n,k} D_{m,l}^* \sqrt{\lambda_n^1\lambda_m^2}|\rho_n^1\rangle\langle\rho_m^2| \qquad (31)\\ & + \sum_{nm}\sum_{lk} \langle s1,k|Q|s2,l\rangle C_{n,k}^* D_{m,l} \sqrt{\lambda_n^1\lambda_m^2}|\rho_m^2\rangle\langle\rho_n^1|, \quad \forall Q \in \mathcal{L}(\mathcal{H}_S).\end{aligned}$$

C_3: *There exists a density operator ρ_2 on $\mathcal{H}_T$ such that $P_{t2}\rho_2 P_{t2} = \rho_2$, and a trace preserving CPM Φ_2 with source $\mathcal{H}_{s2}$ and target $\mathcal{H}_{t2}$ such that*

$$\Phi(Q) = \rho_2 \operatorname{Tr}(P_{s1}Q) + \Phi_2(Q), \quad \forall Q \in \mathcal{L}(\mathcal{H}_S). \qquad (32)$$

C_4: *There exists a density operator ρ_1 on $\mathcal{H}_T$ such that $P_{t1}\rho_1 P_{t1} = \rho_1$, and a trace preserving CPM Φ_1 with source $\mathcal{H}_{s1}$ and target $\mathcal{H}_{t1}$ such that*

$$\Phi(Q) = \rho_1 \operatorname{Tr}(P_{s2}Q) + \Phi_1(Q), \quad \forall Q \in \mathcal{L}(\mathcal{H}_S). \qquad (33)$$

If a trace preserving CPM Φ fulfils the conditions of one of these cases, then Φ is subspace local from $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$ to $(\mathcal{H}_{t1}, \mathcal{H}_{t2})$.

proof. Since Φ is subspace local there exists, according to proposition 1, a product channel $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$ of trace preserving CPMs, which has Φ as its 1-restriction, and is such that lemma 5 is applicable. The strategy of this proof is the following. For each of the four cases in lemma 5 it is shown that one of the cases C_1, C_2, C_3, or C_4 is implied. This is done by taking the 1-restriction of $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$ for each case in lemma 5. Moreover, for each of the cases C_j the opposite implication, stated in the end of the proposition, is proved.

We begin with the first case of lemma 5. It is straightforward to convince oneself that the first case in lemma 5 corresponds to LSP channels. LSP channels are SL channels, hence the opposite implication stated in the proposition is proved.

Consider the second case in lemma 5. Since $\widetilde{\Phi}_1$ is SP from $(F_0(\mathcal{H}_{s1}), F_1(\mathcal{H}_{s1}))$ to $(F_1(\mathcal{H}_{t1}), F_0(\mathcal{H}_{t1}))$, it is an SP gluing of two trace preserving CPMs $\widetilde{\Delta}_a$, $\widetilde{\Delta}_b$ (see [2] proposition 1). $\widetilde{\Delta}_a$ with source $F_0(\mathcal{H}_{s1})$ and target $F_1(\mathcal{H}_{t1})$, $\widetilde{\Delta}_b$ with source $F_1(\mathcal{H}_{s1})$ and target $F_0(\mathcal{H}_{t1})$. Note that the source space $F_0(\mathcal{H}_{s1})$ of $\widetilde{\Delta}_a$ is one-dimensional. One can realize that if such a map is to be a trace preserving

CPM then it can be written $\Delta_a(\widetilde{Q}_1) = \widetilde{\rho}_1\langle\widetilde{0}_{s1}|\widetilde{Q}_1|\widetilde{0}_{s1}\rangle$ for all $\widetilde{Q}_1 \in \mathcal{L}(F_0(\mathcal{H}_{s1}))$, where $\widetilde{\rho}_1$ is a density operator on $F_1(\mathcal{H}_{t1})$. $\widetilde{\Delta}_b$ on the other hand, has the one-dimensional target space $F_0(\mathcal{H}_{t1})$. The only possibility for such a map to be a trace preserving CPM is $\Delta_b(\widetilde{Q}_1) = |\widetilde{0}_{t1}\rangle\langle\widetilde{0}_{t1}|\,\mathrm{Tr}(\widetilde{Q}_1)$ for all $\widetilde{Q}_1 \in \mathcal{L}(F_1(\mathcal{H}_{s1}))$. Note that $K = \dim(F_1(\mathcal{H}_{s1})) = \dim(\mathcal{H}_{s1})$. Let $\{\lambda_n^1\}_{n=1}^N$ be the non-zero eigenvalues and $\{|\widetilde{\rho}_n^1\rangle\}_{n=1}^N$ a corresponding set of orthonormal eigenvectors of $\widetilde{\rho}_1$. Note that $N \leq \dim(F_1(\mathcal{H}_1)) = \dim(\mathcal{H}_{t1})$. The CPM $\widetilde{\Delta}_a$ has a linearly independent Kraus representation [1] on the form $\{\sqrt{\lambda_n^1}|\widetilde{\rho}_n^1\rangle\langle\widetilde{0}_{s1}|\}_n$. Likewise $\{|\widetilde{0}_{t1}\rangle\langle\widetilde{s1},k|\}_{k=1}^K$ is a linearly independent Kraus representation of Δ_b. Since $\widetilde{\Phi}_1$ is an SP gluing of $\widetilde{\Delta}_a$ and Δ_b it follows, by proposition 3 in [2], that $\widetilde{\Phi}_1$ can be written as

$$\begin{aligned}\widetilde{\Phi}_1(\widetilde{Q}_1) = {} & \widetilde{\rho}_1\langle\widetilde{0}_{s1}|\widetilde{Q}_1|\widetilde{0}_{s1}\rangle + |\widetilde{0}_{t1}\rangle\langle\widetilde{0}_{t1}|\,\mathrm{Tr}(P_{\widetilde{s1}:1}\widetilde{Q}_1)\\ & + \sum_{nk} C_{n,k}\sqrt{\lambda_n^1}|\widetilde{\rho}_n^1\rangle\langle\widetilde{0}_{s1}|\widetilde{Q}_1|\widetilde{s1},k\rangle\langle\widetilde{0}_{t1}|\\ & + \sum_{nk} C^*_{n,k}|\widetilde{0}_{t1}\rangle\langle\widetilde{s1},k|\widetilde{Q}_1|\widetilde{0}_{s1}\rangle\langle\widetilde{\rho}_n^1|\sqrt{\lambda_n^1}, \quad \forall\widetilde{Q}_1 \in \mathcal{L}(F_{01}(\mathcal{H}_{s1})), \end{aligned} \tag{34}$$

where the matrix $C = [C_{n,k}]_{n,k}$ fulfils the condition $CC^\dagger \leq I_N$. An analogous reasoning, but applied to $\widetilde{\Phi}_2$, leads to

$$\begin{aligned}\widetilde{\Phi}_2(\widetilde{Q}_2) = {} & \widetilde{\rho}_2\langle\widetilde{0}_{s2}|\widetilde{Q}_2|\widetilde{0}_{s2}\rangle + |\widetilde{0}_{t2}\rangle\langle\widetilde{0}_{t2}|\,\mathrm{Tr}(P_{\widetilde{s2}:1}\widetilde{Q}_2)\\ & + \sum_{ml} D_{m,l}\sqrt{\lambda_m^2}|\widetilde{\rho}_m^2\rangle\langle\widetilde{0}_{s2}|\widetilde{Q}_2|\widetilde{s2},l\rangle\langle\widetilde{0}_{t2}|\\ & + \sum_{ml} D^*_{m,l}|\widetilde{0}_{t2}\rangle\langle\widetilde{s2},l|\widetilde{Q}_2|\widetilde{0}_{s2}\rangle\langle\widetilde{\rho}_m^2|\sqrt{\lambda_m^2}, \quad \forall\widetilde{Q}_2 \in \mathcal{L}(F_{01}(\mathcal{H}_{s2})), \end{aligned} \tag{35}$$

where $\widetilde{\rho}_2$ is a density operator on $F_1(\mathcal{H}_{t2})$, $\{\lambda_m^2\}_{m=1}^M$ the non-zero eigenvalues and $\{|\widetilde{\rho}_m^2\rangle\}_{m=1}^M$ a corresponding set of orthonormal eigenvectors of $\widetilde{\rho}_2$. The matrix $D = [D_{m,l}]_{m,l=1}^{M,L}$ fulfils the condition $DD^\dagger \leq I_M$. The 1-restriction of $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$ can be calculated using (34), (35), and (1). One may verify that the result of the 1-restriction is case C_2, with

$$\begin{aligned} &\rho_1 = M_{t1}^\dagger\widetilde{\rho}_1 M_{t1}, && \rho_2 = M_{t2}^\dagger\widetilde{\rho}_2 M_{t2},\\ &|\rho_n^1\rangle = M_{t1}^\dagger|\widetilde{\rho}_n^1\rangle, && |\rho_m^2\rangle = M_{t2}^\dagger|\widetilde{\rho}_m^2\rangle,\\ &|s1,k\rangle = M_{s1}^\dagger|\widetilde{s1},k\rangle, && |s2,l\rangle = M_{s2}^\dagger|\widetilde{s2},l\rangle. \end{aligned} \tag{36}$$

To show that every channel of type C_2 is SL, it has to be shown that every channel on the form (31) is an SL channel. Define $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ by equations (34) and (35), with

$$\begin{aligned} &\widetilde{\rho}_1 = M_{t1}\rho_1 M_{t1}^\dagger, && \widetilde{\rho}_2 = M_{t2}\rho_2 M_{t2}^\dagger,\\ &|\widetilde{\rho}_n^1\rangle = M_{t1}|\rho_n^1\rangle, && |\widetilde{\rho}_m^2\rangle = M_{t2}|\rho_m^2\rangle,\\ &|\widetilde{s1},k\rangle = M_{s1}|s1,k\rangle, && |\widetilde{s2},l\rangle = M_{s2}|s2,l\rangle. \end{aligned} \tag{37}$$

By the assumed properties of ρ_1 and ρ_2 it follows that $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$ is trace preserving and has Φ as its 1-restriction.

We next turn to the third case of lemma 5. As a partial step in the calculation of the 1-restriction one can use the following:

$$\begin{aligned}\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2(|\widetilde{0}_{s1}\rangle\langle\widetilde{0}_{s1}| \otimes |\widetilde{s2},l\rangle\langle\widetilde{s2},l'|) &= |\widetilde{0}_{t1}\rangle\langle\widetilde{0}_{t1}| \otimes \widetilde{\Phi}_2(|\widetilde{s2},l\rangle\langle\widetilde{s2},l'|),\\ \widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2(|\widetilde{s1},k\rangle\langle\widetilde{s1},k'| \otimes |\widetilde{0}_{s2}\rangle\langle\widetilde{0}_{s2}|) &= \delta_{kk'}|\widetilde{0}_{t1}\rangle\langle\widetilde{0}_{t1}| \otimes \widetilde{\Phi}_2(|\widetilde{0}_{s2}\rangle\langle\widetilde{0}_{s2}|),\end{aligned}$$

$\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2(|\widetilde{s1},k\rangle\langle\widetilde{0}_{s1}| \otimes |\widetilde{0}_{s2}\rangle\langle\widetilde{s2},l|) = 0, \quad \widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2(|\widetilde{0}_{s1}\rangle\langle\widetilde{s1},k'| \otimes |\widetilde{s2},l\rangle\langle\widetilde{0}_{s2}|) = 0.$

By taking the 1-restriction, case C_3 is found, with $\rho_2 = M_T^\dagger|\widetilde{0}_{t1}\rangle\langle\widetilde{0}_{t1}| \otimes \widetilde{\Phi}_2(|\widetilde{0}_{s2}\rangle\langle\widetilde{0}_{s2}|)M_T$, and Φ_2 defined as $\Phi_2(Q) = M_{t2}^\dagger\widetilde{\Phi}_2(M_{s2}QM_{s2}^\dagger)M_{t2}$. Note that Φ_2, regarded as having source $\mathcal{H}_{s2}$ and target $\mathcal{H}_{t2}$, is trace preserving. The operator ρ_2 is a density operator since $\widetilde{\Phi}_2$ is trace preserving and $P_{\widetilde{t2}:1}\widetilde{\Phi}_2(|\widetilde{0}_{s2}\rangle\langle\widetilde{0}_{s2}|)P_{\widetilde{t2}:1} = \widetilde{\Phi}_2(|\widetilde{0}_{s2}\rangle\langle\widetilde{0}_{s2}|)$. From this also follows that $P_{t2}\rho_2P_{t2} = \rho_2$.

To show that CPMs of the form (32) are SL channels, define $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ by

$$\begin{aligned}
&\widetilde{\Phi}_1(\widetilde{Q}_1) = |\widetilde{0}_{t1}\rangle\langle\widetilde{0}_{t1}|\operatorname{Tr}(\widetilde{Q}_1), \quad \forall\widetilde{Q}_1 \in \mathcal{L}(F_{01}(\mathcal{H}_{s1})),\\
&\widetilde{\Phi}_2(\widetilde{Q}_2) = M_{t2}\Phi_2(M_{s2}^\dagger\widetilde{Q}_2M_{s2})M_{t2}^\dagger + \widetilde{\rho}_2\operatorname{Tr}(P_{\widetilde{s2}:0}\widetilde{Q}_2), \quad \forall\widetilde{Q}_2 \in \mathcal{L}(F_{01}(\mathcal{H}_{s2})),
\end{aligned} \tag{38}$$

where $\widetilde{\rho}_2 = M_{t2}\rho_2M_{t2}^\dagger$. One can show that $\widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2$ is trace preserving and has Φ as its 1-restriction. (Hence it respects 1-states. See lemma 2). Hence, Φ is an SL channel.

By reasoning analogous to the third case, one finds that the fourth case of lemma 5 implies C_4. Likewise, channels of the form (33) can be shown to be SL channels. □

As a quite direct consequence of proposition 4 we have the following corollary.

Corollary 1 *Let Φ be an SL channel from $(\mathcal{H}_{s1},\mathcal{H}_{s2})$ to $(\mathcal{H}_{t1},\mathcal{H}_{t2})$. The following gives necessary and sufficient conditions for Φ to belong to one of the four classes of SL channels given in proposition (4).*

C_1: $\operatorname{Tr}(P_{t1}\Phi(Q)) = \operatorname{Tr}(P_{s1}Q), \quad \forall Q \in \mathcal{L}(\mathcal{H}_S)$.

C_2: $\operatorname{Tr}(P_{t1}\Phi(Q)) = \operatorname{Tr}(P_{s2}Q), \quad \forall Q \in \mathcal{L}(\mathcal{H}_S)$.

C_3: $\operatorname{Tr}(P_{t2}\Phi(Q)) = \operatorname{Tr}(Q), \quad \forall Q \in \mathcal{L}(\mathcal{H}_S)$.

C_4: $\operatorname{Tr}(P_{t1}\Phi(Q)) = \operatorname{Tr}(Q), \quad \forall Q \in \mathcal{L}(\mathcal{H}_S)$.

In words this corollary says that the four types of SL channels can be classified according to how they handle the probability weights on the two locations. The LSP channels preserve the probability weight on each location. Class C_2 swaps the probability weights. Class C_3 and C_4 concentrate the probability into one of the two locations. Hence the LSP channels are the only channels that do not redistribute the probability weights between the two locations.

At first sight it may seem surprising that a channel which we claim to act locally, should have the power to redistribute the probability weights between the two locations. Seen from the point of view of second quantization however, this is not very surprising. Take the channels of type 3 as an example. At location 1 the channel $\widetilde{\Phi}_1$ acts by returning the vacuum state $|\widetilde{0}_{t1}\rangle$, no matter the input state. At the other location $\widetilde{\Phi}_2$ is acting as some channel on the single-particle states, preserving single-particle states as single-particle states, but maps the vacuum state to a fixed single-particle state. As seen, the local particle number is not conserved, but the removal of the particle at location 1 is compensated for at location 2, where the vacuum state is mapped to a single-particle state. The two channels act independently of each other, but together they act as if a particle was 'transferred' from location 1 to location 2.

The following proposition shows that the LSP channels can be characterized as the subspace local SP channels.

Proposition 5 *The intersection of the set of SP channels from $(\mathcal{H}_{s1},\mathcal{H}_{s2})$ to $(\mathcal{H}_{t1},\mathcal{H}_{t2})$ and the set of SL channels from $(\mathcal{H}_{s1},\mathcal{H}_{s2})$ to $(\mathcal{H}_{t1},\mathcal{H}_{t2})$, is the set of LSP channels from $(\mathcal{H}_{s1},\mathcal{H}_{s2})$ to $(\mathcal{H}_{t1},\mathcal{H}_{t2})$.*

proof. We first prove that $SP \supset LSP$. By corollary 1 every LSP channel fulfils $\mathrm{Tr}(P_{t1}\Phi(Q)) = \mathrm{Tr}(P_{s1}Q)$ for all $Q \in \mathcal{L}(\mathcal{H}_S)$. Hence, by proposition 1 in [2] it follows that Φ is SP. By construction $SL \supset LSP$. By combining these inclusions it follows that $SL \cap SP \supset LSP$. It remains to show the opposite inclusion. It is sufficient to show that the last three families of SL channels, described in proposition 4, cannot be SP channels. By combining proposition 4 in [1] and corollary 1 one finds that none of the last three types of SL channels can be SP. □

The following proposition provides a 'composition table' for SL channels and shows that a composition of two SL channels is again an SL channel, which fits with the intuitive notion of locally realizable channels. Let $\mathcal{H}_R = \mathcal{H}_{r1} \oplus \mathcal{H}_{r2}$ be finite-dimensional. Assume that $\mathcal{H}_{r1}$ and $\mathcal{H}_{r2}$ are at least one-dimensional.

Proposition 6 *If a CPM Φ_a is SL from $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$ to $(\mathcal{H}_{t1}, \mathcal{H}_{t2})$ and if a CPM Φ_b is SL from $(\mathcal{H}_{t1}, \mathcal{H}_{t2})$ to $(\mathcal{H}_{r1}, \mathcal{H}_{r2})$, then $\Phi_b \circ \Phi_a$ is SL from $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$ to $(\mathcal{H}_{r1}, \mathcal{H}_{r2})$.*

If moreover Φ_a belongs to class C_i and Φ_b belongs to class C_j, then $\Phi_b \circ \Phi_a$ belongs to class C_k according to the following rules:

$$C_1 \circ C_j \subset C_j, \quad C_j \circ C_1 \subset C_j \quad j = 1, \ldots, 4 \tag{39}$$

$$C_i \circ C_j \subset C_i \quad i = 3, 4 \quad j = 1, \ldots, 4 \tag{40}$$

$$C_2 \circ C_2 \subset C_1, \quad C_2 \circ C_3 \subset C_4, \quad C_2 \circ C_4 \subset C_3 \tag{41}$$

proof. Since Φ_a and Φ_b are SL there exists trace preserving product channels $\widetilde{\Phi}_{a1} \otimes \widetilde{\Phi}_{a2}$ and $\widetilde{\Phi}_{b1} \otimes \widetilde{\Phi}_{b2}$, with Φ_a respectively Φ_b as 1-restrictions. Hence $\Phi_a(Q) = M_T^\dagger[\widetilde{\Phi}_{a1} \otimes \widetilde{\Phi}_{a2}](M_S Q M_S^\dagger) M_T$ and $\Phi_b(Q) = M_R^\dagger[\widetilde{\Phi}_{b1} \otimes \widetilde{\Phi}_{b2}](M_T Q M_T^\dagger) M_R$. Using $M_T M_T^\dagger = P_{\widetilde{T}:1}$ it follows that

$$\begin{aligned}\Phi_b \circ \Phi_a(Q) &= M_R^\dagger[\widetilde{\Phi}_{b1} \otimes \widetilde{\Phi}_{b2}](P_{\widetilde{T}:1}[\widetilde{\Phi}_{a1} \otimes \widetilde{\Phi}_{a2}](M_S Q M_S^\dagger) P_{\widetilde{T}:1}) M_R \\ &= M_R^\dagger[(\widetilde{\Phi}_{b1} \circ \widetilde{\Phi}_{a1}) \otimes (\widetilde{\Phi}_{b2} \circ \widetilde{\Phi}_{a1})](M_S Q M_S^\dagger) M_R.\end{aligned} \tag{42}$$

The last equality holds according to lemma 1, since $\widetilde{\Phi}_{a1} \otimes \widetilde{\Phi}_{a2}$ respects 1-states. Hence $(\widetilde{\Phi}_{b1} \circ \widetilde{\Phi}_{a1}) \otimes (\widetilde{\Phi}_{b2} \circ \widetilde{\Phi}_{a1})$ has $\Phi_b \circ \Phi_a$ as 1-restriction. Hence, $\Phi_b \circ \Phi_a$ is an SL channel. The rest of the proposition follows from corollary 1. □

6. Discussion

Here we discuss some conceptual aspects of the SL channels in general. Imagine for a moment that we live in a universe where all particles are distinguishable and that none of these can be annihilated and recreated again. Suppose we have two boxes and one particle. These two boxes are located far away from each other and the particle can be in any state of superposition or mixture of being localized in the boxes. Consider general operations on the state of this particle. Hence, we are considering channels with identical source and target space $\mathcal{H}_{s1} \oplus \mathcal{H}_{s2}$, where $\mathcal{H}_{s1}$, $\mathcal{H}_{s2}$ represent the pure localized states on the two boxes. What subspace local channels on $(\mathcal{H}_{s1}, \mathcal{H}_{s2})$, of this type, are possible to perform subspace locally? With the restrictions assumed for this toy universe and with the assumed setup, only the LSP channels are possible to perform subspace locally; the reason being that the other three classes of SL channels do not conserve the local particle number. If there are identical particles, a particle can be removed (or perhaps annihilated) from the first box, and an identical particle can be inserted (or perhaps created) at the second box. In this case, however, there is no identical particle to be inserted or created in the other box. Hence, the only possibility to realize such a channel would be to physically transport the particle from one box to the other, which cannot reasonably be called a 'local' operation. Far from being clearcut, this example suggests that there is a connection between questions of locality and non-locality of quantum channels and the existence of identical particles. At least from the point of view presented here, a universe with identical particles seems more 'allowing' than a universe without. Whether or not this reflection carry any substance in a wider perspective remains to see.

From the discussion above one may be tempted to make the conclusion that in a universe with only distinguishable particles, the only SL channels possible to realize are the LSP channels. This is however a too rapid conclusion, as the following examples show. Consider again the two-box system described above, but this time with two distinguishable particles instead of one. We call them the S-particle and the T-particle. The state space of the S particle on the two boxes is $\mathcal{H}_S = \mathcal{H}_{s1} \oplus \mathcal{H}_{s2}$, where $\mathcal{H}_{s1}$ represents the pure states of particle S localized in box 1 and so on. Likewise $\mathcal{H}_{t1}$ and $\mathcal{H}_{t2}$ represents the pure localized states of the T-particle in the two boxes. We let the T-particle be in some fixed initial state ρ_T and ask how the state of the S-particle affects the state of the T-particle, if the particles initially are in a product state $\rho_S \otimes \rho_T$. To create examples of SL channels of type C_4, we let the T-particle be localized in box 1 ($P_{t1}\rho_T P_{t1} = \rho_T$). The most simple example of a SL mapping of type C_4 is the extreme case of no interaction between the two particles. In that case one obtains the map $\Phi(\rho_S) = Tr_S(\rho_S \otimes \rho_T) = \rho_T$. It is possible, however, to create a bit less trivial examples. We assume the S and T particle to interact only if they occupy the same box. It seems reasonable to assume an Hamiltonian on the form (for the sake of simplicity we only consider interaction and no Hamiltonians for the particles themselves)

$$\begin{aligned} H = & \sum_{k,k',n,n'} H_{k,n;k',n'} |s1,k\rangle\langle s1,k'| \otimes |t1,n\rangle\langle t1,n'| \\ & + \sum_{l,l',m,m'} H_{l,m;l',m'} |s2,l\rangle\langle s2,l'| \otimes |t2,m\rangle\langle t2,m'| \end{aligned} \quad (43)$$

Again we assume the initial state to be a product state and let the T-particle be in a fixed localized state, and ask how the initial S-state affects the T-state after some fixed (but arbitrary) elapse of time t. The following channel answers this question: $\Phi(\rho_S) = \mathrm{Tr}_S(e^{-itH}\rho_S \otimes \rho_T e^{itH})$, for all density operators ρ_S on $\mathcal{H}_S$, where we let $\hbar = 1$. It is possible to deduce that Φ is an SL channel of type C_4, by using $P_{t1}\rho_T P_{t1} = \rho_T$ and the structure of 43. Note that this is true irrespective of the choice of time t. Moreover, note that the initial state ρ_T is chosen to be local, hence in some sense one does not expect it to be a 'non-local resource'. As seen neither annihilation nor creation of particles has been assumed in this construction.

We can conclude that, if we we are considering state changes on one and the same system, only the LSP channels are locally realizable in this 'gedanken universe', while if we consider mappings from one system to another, also other SL channels are locally realizable. Hence in the latter case there is, in some sense, more freedom. This suggest that the set of channels which are locally realizable depends on the specific context.

Before ending this section we mention one further aspect on the relation between the LSP channels and the rest of the SL channels. One intuitively reasonable requirement for a operation to be local with respect to two locations, is that it should be possible to compose it out of two operations: one which operates on location 1, while doing 'nothing' on location 2, and the other operating on the other location while doing nothing on the first. In case of locality with respect to tensor product decomposition this is trivially satisfied, since $\Phi_1\otimes\Phi_2 = (I_1\otimes\Phi_2)\circ(\Phi_1\otimes I_2)$. By the very construction of the here proposed definition of subspace locality, all SL operations can be decomposed in this way, in terms of operations on the second quantized spaces. This since any SL channel corresponds to a product channel $\widetilde{\Phi}_1\otimes\widetilde{\Phi}_2$ on the second quantized spaces. However, the LSP channels do admit a much simpler decomposition, directly in terms of the first quantized spaces. A reasonable interpretation of "operating on location 1 and doing nothing on location 2", is to have a channel which is a trace preserving gluing [2] of a channel on location 1 and an identity CPM on location 2.

Proposition 7 *A channel Φ is LSP from $(\mathcal{H}_{s1},\mathcal{H}_{s2})$ to $(\mathcal{H}_{r1},\mathcal{H}_{r2})$, if and only if there exist channels Φ_a and Φ_b such that $\Phi = \Phi_b\circ\Phi_a$, where Φ_a is a trace preserving gluing of a channel with source $\mathcal{H}_{s1}$ and target $\mathcal{H}_{t1}$, and the identity CPM with source and target $\mathcal{H}_{s2}$, and where Φ_b is a trace preserving gluing of a channel with source $\mathcal{H}_{s2}$ and target $\mathcal{H}_{t2}$, and the identity CPM with source and target $\mathcal{H}_{t1}$*

Note that there exists an equivalent formulation of this proposition, where the ordering of the operations is reversed.

proof. We begin with the "if" part. Φ_a and Φ_b are both LSP channels since a trace preserving gluing of a trace preserving CPM and an identity CPM necessarily is an LSP channel (see proposition 7 in [2]). By proposition 6 it follows that $\Phi = \Phi_b\circ\Phi_a$ is LSP.

For the "only if" part, let $\{V_n\}_n$, V, $\{W_m\}_m$, and W, be the operators in proposition 2, with respect to the LSP channel Φ. Let $\Phi_a(Q) = \sum_n V_nQV_n^\dagger + P_{s2}QP_{s2} + VQP_{s2} + P_{s2}QV^\dagger$, for all $Q\in\mathcal{L}(\mathcal{H}_S)$. Let $\Phi_b(Q) = P_{t1}QP_{t1} + \sum_m W_mQW_m^\dagger + P_{t1}QW^\dagger + WQP_{t1}$, for all $Q\in\mathcal{L}(\mathcal{H}_{t1}\oplus\mathcal{H}_{s2})$. Clearly $\Phi = \Phi_b\circ\Phi_a$ and one can control that each of Φ_a and Φ_b are gluings of channels and identity CPMs, as stated in the proposition. $\square$

There are more questions that may be raised on the nature of the concept of subspace locality and the here proposed way to formalize it. More investigation is needed to settle which is the most preferable definition of subspace locality in different contexts. Some more aspects of subspace locality is discussed in [2].

7. Summary

A definition of *subspace locality* (SL) of quantum channels (trace preserving completely positive maps) is proposed. The purpose of this definition is to formulate conditions which channels have to fulfill, if they are to act 'locally', when the division in locations does not naturally correspond to a tensor product decomposition of the Hilbert space, but rather a decomposition into an orthogonal sum. One example of such a system is a particle in a two-path interferometer, where the total Hilbert space of the particle can be decomposed into an orthogonal sum of two Hilbert spaces, each representing pure localized states in one of the paths. Given a quantum channel acting on the state of a particle in the two paths, we wish to find some condition which the channel has to fulfill, if it is to act 'locally' on each path.

The here proposed definition of subspace locality is stated in terms of occupation number representations of second quantizations of the involved Hilbert spaces. It is used that in the occupation number representation, a second quantization of an orthogonal sum of two subspaces, is equivalent to a tensor product of the second quantizations of each of the two subspaces. With respect to this tensor product decomposition, the 'usual' definition of a locally acting channel as a product channel, is used. The consequences of this choice of definition is investigated.

Under the restricting assumption that the first quantized state spaces are all finite-dimensional, the here proposed definition of subspace locality of quantum channels is reformulated in the original first quantized state spaces. This gives expressions which makes it possible to explicitly generate all subspace local channels. Moreover, it is shown that the set of all SL channels decomposes into four disjoint families. One of these families, called *local subspace preserving* (LSP), is shown to be the intersection between the set of SL channels and the set of subspace preserving channels [1]. Proposition 2 gives an explicit construction of all the LSP channels. Proposition 4 provides explicit expressions for all the four families of SL channels. For a subclass of the LSP channels a special form of construction in terms of a joint unitary evolution with an ancilla system is proved.

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